General Relativity | Week 4

Definition: A connection on M is a map $\nabla : \Gamma(M) \times \Gamma(M) \longrightarrow \Gamma(M)$

which is

1) $C^\infty(M)$-linear in 1^{st} argument:

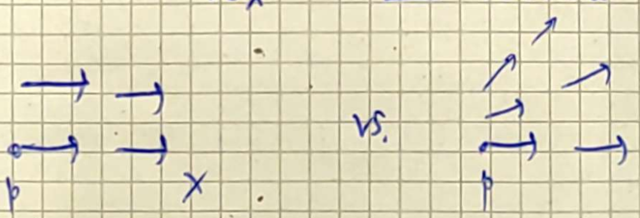
$$\nabla_{X_1 + f \cdot X_2} Y = \nabla_{X_1} Y + f \cdot \nabla_{X_2} Y \, , \quad f \in C^\infty(M)$$

2) $\mathbb{R}$-linear in 2^{nd} argument:

$$\nabla_X (Y_1 + \lambda Y_2) = \nabla_X Y_1 + \lambda \nabla_X Y_2 \, , \quad \lambda \in \mathbb{R}$$

3) Leibniz rule for 2^{nd} argument: $\nabla_X (f \cdot Y) = X(f) \cdot Y + f \cdot \nabla_X Y$.

Note:
- $\nabla_X Y|_p$: Directional derivative of Y at p in the direction $X|_p$
- ∇Y : $(1,1)$-tensor field.
- The Lie derivative $\mathcal{L}_X Y$ is not tensorial in X (depends on ∂X)



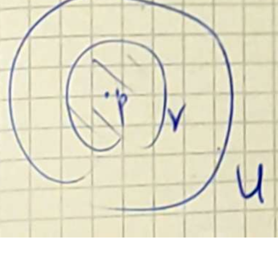
The connection is "locally defined":

Lem: $Y \in \Gamma(M)$, $Y|_U = 0$ on an open set $U \subseteq M$. Then

$\forall p \in U$, $X \in \Gamma(M)$: $\nabla_X Y|_p = 0$

Proof: Let $f : M \to \mathbb{R}$ be such that $f = 1$ in a neighborhood $V \subset U$ of p, $f = 0$ on $M \setminus U$.

Then $f \cdot Y = 0$ everywhere:



$$\Rightarrow \quad 0 = \nabla_X (f \cdot Y)|_p = \cancel{X(f)|_p} \cdot \underbrace{Y|_p}_{=0} + \underbrace{f(p)}_{=1} \cdot \nabla_X Y|_p \, . \quad \square$$

So: We can talk about ∇ locally around points

(i.e. for vector fields defined only on open sets, e.g. coordinate vector fields)

Christoffel symbols: In any local coordinate system $(x^0, \dots, x^n)$,

$$\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial x^\gamma} \quad \text{or} \quad \Gamma^\gamma_{\alpha\beta} = dx^\gamma \left(\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} \right)$$

Γ: Not a tensor field (depends on the coordinate system)

(But $\nabla - \bar{\nabla}$, or $\Gamma - \bar{\Gamma}$, is a tensor field).

Note: $$\boxed{(\nabla_X Y)^k = X^a \partial_a Y^k + \Gamma^k_{a\beta} X^a Y^\beta}$$

Example: Flat connection on $\mathbb{R}^n$: In Cartesian coordinates,

$$(\nabla_X Y)^k = X^a \partial_a Y^k$$ (So $\Gamma^k_{ij} = 0$ - not true in other coordinates, e.g. polar coordinates)

Def: Torsion: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

(Note: $\nabla^*_X Y = \nabla_Y X + [X,Y]$: "Dual" connection)

- Antisymmetric (1,2)-tensor field
- In local coordinates: $T^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}$

∇ is torsion free iff $T \equiv 0 \iff \Gamma^k_{ij} = \Gamma^k_{ji} \iff \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$

Since the difference is a tensor, it's enough if the equality holds in one coordinate system.

We can extend ∇ to act on <u>tensor fields</u> by the requirement that

- $\operatorname{tr}(\nabla_X f) = \nabla_X(\operatorname{tr} f)$
- $\nabla_X(f \otimes h) = \nabla_X f \otimes h + f \otimes \nabla_X h$

$\left(\nabla : \Gamma(M) \times \mathcal{T}^{(k,l)}(M) \to \mathcal{T}^{(k,l)}(M)\right)$

For instance, from the above two properties: For 1-forms

$$\boxed{(\nabla_X \omega)_\alpha = X^\beta \partial_\beta \omega_\alpha - \Gamma^\lambda_{\beta\alpha} X^\beta \omega_\lambda}$$

For instance, Hessian of a function:

$$(\nabla df)_{\alpha\beta} = \partial^2_{\alpha\beta} f - \Gamma^k_{\alpha\beta} \partial_k f.$$

<u>Levi-Civita connection</u>:

Let (M,g) be a Lorentzian manifold. There exists a <u>unique</u> connection ∇ on M such that:

1) ∇ is <u>torsion free</u>
2) ∇ respects the metric:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$(\Leftrightarrow \nabla g = 0)$$

<u>Construction</u>: In local coordinates: $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ (torsion free)

and $$\partial_\alpha g_{\beta\gamma} = \partial_\alpha(g(\partial_\beta, \partial_\gamma)) = g(\nabla_{\partial_\alpha}\partial_\beta, \partial_\gamma) + g(\partial_\beta, \nabla_{\partial_\alpha}\partial_\gamma)$$

$$= \Gamma^k_{\alpha\beta} g_{k\gamma} + \Gamma^k_{\alpha\gamma} g_{k\beta}$$

Permutting α, β, γ:

$$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta} = 2\Gamma^k_{\alpha\beta} g_{k\gamma}$$

$$\Rightarrow \boxed{\Gamma^k_{\alpha\beta} = \frac{1}{2} g^{k\gamma}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}\right)}$$

Remarks: • Canonical connection

• If we drop the torsion-free condition: Non-unique (excep in 2d)

• "Commutes" with functions of g

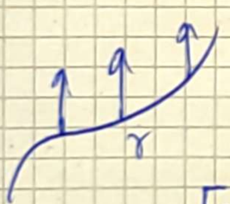
eg the volume form, the musical isomorphisms

i.e. $(\nabla X)_\flat = \nabla(X_\flat)$ $\quad (X_\flat = tr(g \otimes X))$

So will write $\nabla_a X_\beta$ without specifying if the musical isomorphism was applied before or after differentiation.

Parallel transport:

Let $\gamma : (a,b) \to M$ be a smooth curve.



I can consider vector fields along γ

$X \in \Gamma_\gamma \; : \; t \longrightarrow X_t \in T_{\gamma(t)} M$

Ex: $\dot{\gamma}$

Note: At points of self-intersections of γ: $X \in \Gamma_\gamma$ cannot be extended to $\Gamma(M)$

But always. If I shrink the domain of γ around t, so that no self intersection takes place: I can extend (non-uniquely) $X \in \Gamma_\gamma$ to $X \in \Gamma(M)$.

So I can talk about $\nabla_{\dot\gamma} X$

(For any such extension: $\nabla_{\dot\gamma} X |_{\gamma(t)}$ is the same)

Proof: In local coordinates around $\gamma(t)$:

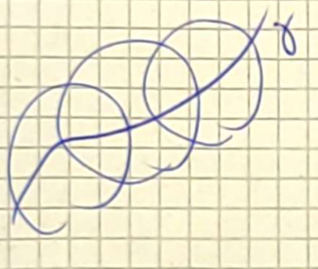
$$(\nabla_{\dot\gamma} X)^k = \underbrace{\dot\gamma^r \partial_r X^k}_{\dot\gamma(X^k)} + \Gamma^k_{ij}(\gamma(t)) \cdot \dot\gamma^i X^j$$

↖ only depend on values of X^k along γ. ☐

Def: $X \in \Gamma_\gamma$ is parallel transported along γ if $\nabla_{\dot\gamma} X = 0$

$$\forall\ \xi \in T_{\gamma(t_0)} M: \quad \exists!\ X \in \Gamma_\gamma \ \text{ such that } \begin{cases} \nabla_{\dot\gamma} X = 0 \\ X|_{\gamma(t_0)} = \xi \end{cases}$$

Notation: $X|_{\gamma(t_1)} = \mathbb{P}^{(\gamma)}_{t_0 \to t_1} \xi$.



Proof: In local coordinates on $U \cap \gamma$:

$$\begin{cases} \frac{d}{dt} X^k + \Gamma^k_{ij} \dot\gamma^i X^j = 0 \\ X^k|_{t=t_0} = \xi^k \end{cases}$$

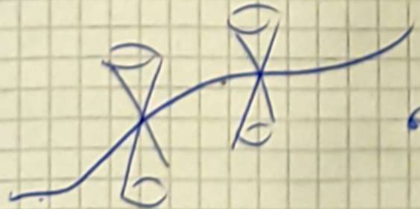
1st order linear ODE

$\Rightarrow$ unique solution along $U \cap \gamma$

Covering the curve with domains of coordinate charts and using uniqueness on the overlaps: I can define X on the whole of γ

□

The parallel transport map $\mathbb{P}^{(\gamma)}_{t_0 \to t_1}: T_{\gamma(t_0)} M \longrightarrow T_{\gamma(t_1)} M$ is a linear isometry: $\nabla_{\dot\gamma} \langle X, Y \rangle = 0$ if X, Y are parallel transported.



$\leftarrow$ $\mathbb{P}^{(\gamma)}$ sends the light cone to the light cone.

Geodesic: Self parallel line: $\nabla_{\dot\gamma} \dot\gamma = 0$

(So if $\dot\gamma(t_0)$ timelike: $\dot\gamma$ everywhere timelike

Similarly for null, spacelike etc)

Since $\nabla_{\dot\gamma} \dot\gamma = 0 \Rightarrow g(\dot\gamma, \dot\gamma) = \text{const}$.

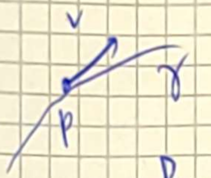
In Minkowski spacetime: geodesics are straight lines.

Thm:

$\forall p \in M$, $\forall v \in T_pM$: $\exists!$ <u>maximal</u> geodesic

$\gamma : (-T_-, T_+) \to M$ with $\gamma(0) = p$, $\dot\gamma(0) = v$

Moreover: In local coordinates, γ^k depends smoothly on p, v.



<u>Proof:</u> In local coordinates, geodesic equation:

$$\ddot{x}^k + \Gamma^k_{ij}(x)\,\dot{x}^i \dot{x}^j = 0$$

So: By the existence, uniqueness and smooth dependence of solutions to the IVP for smooth non-linear ODEs ... $\boxed{\checkmark}$

<u>Remark:</u>
- If γ_1, γ_2 geodesics with $\left.\begin{array}{l}\gamma_1(T) = \gamma_2(T) \\ \dot\gamma_1(T) = \dot\gamma_2(T)\end{array}\right\}$: γ_1, γ_2 agree on their common domain of definition.

- If X is a Killing field:

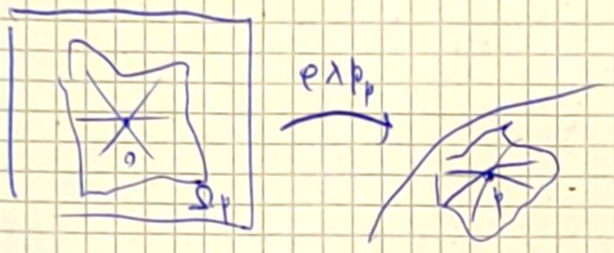
  (Exercise: Show that, $\forall V, W \in \Gamma(M)$,
  $$g(\nabla_V X, W) + g(\nabla_W X, V) = 0)$$

Then $g(X, \dot\gamma) = \text{const}$ along γ. (Exercise).

If I have enough symmetries: I can reduce the study of geodesics to a 1^{st} order problem.

Exponential map:

Let $\Omega_p = \{ v \in T_pM : \gamma_{p,v}$ is defined for $t \in [0,1] \}$ ← geodesic with $\gamma(0)=p$, $\dot\gamma(0)=v$



Define $\exp_p : \Omega_p \to M$

by $\exp_p(v) = \gamma_{p,v}(1)$

- Smooth map (because of the smooth dependence of $\gamma_{p,v}$ on its initial data)

$$\exp_p(tv) = \gamma_{p,v}(t) = \gamma_{p,tv}(1)$$

Lem: $d\exp_p|_{v=0} = id$

Proof: Let $\sigma(t) = tv$ (straight line in T_pM)

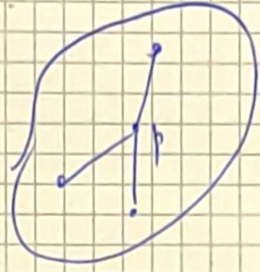
Then $$\underbrace{\frac{d}{dt}\left(\exp_p(\sigma(t))\right)}_{= \dot\gamma_{p,v}(t)} = d\exp_p|_{\sigma(t)}(\dot\sigma(t)) = d\exp_p|_{\sigma(t)}(v)$$

At $t=0$: $\dot\gamma_{p,v}(0) = v \Rightarrow d\exp_p|_0(v) = v$ $\boxed{}$

By the implicit function theorem: $\exp_p$ is a local diffeomorphism on a neighborhood of $0 \in T_pM$.

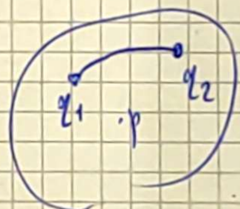
Def: A neighborhood $U \subseteq M$ around p which is the diffeomorphic image through $\exp_p$ of a star-shaped domain in T_pM: normal neighborhood of p.

Normal neighborhood:



← Every point in U is connected to p via a geodesic segment.

- Convex neighborhood of p: If $\forall\, q_1, q_2 \in U$, they are connected by a geodesic segment inside U



Can be constructed via the intersection of normal neighborhoods of points near p

(The implicit function theorem provides uniform bounds for the coordinate size of these normal neighborhoods for points close enough to p).

Normal coordinates around p: (on a normal neighborhood)

Choose an orthonormal frame $\{e_0, \dots, e_n\}$ at T_pM

So $\forall v \in T_pM$: $\quad v = v^a e_a$

Define coordinates $(x^0, \dots, x^n)$ around p so that

$$x^a\left(\exp_p(v)\right) = v^a \iff \left(\exp_p^{-1}(q)\right)^a = x^a(q)$$

In these coordinates: $\exp_p : U \subseteq T_pM \to M$ looks like the identity map.

And if $\gamma_{p,v}(t) = \exp_p(tv)$, then $\gamma^k_{p,v}(t) = t \cdot v^k$

Proposition: Let $(x^0, \dots, x^n)$ be normal coordinates around p.

Then $g_{\alpha\beta}(0) = \eta_{\alpha\beta}$, $\partial_\alpha g_{\beta\gamma}(0) = 0$, $\Gamma^\alpha_{\beta\gamma}(0) = 0$

(but, in general, $\partial^2_{\alpha\beta} g_{\gamma\delta}(0) \neq 0$; this will be related to the curvature tensor of g)